Extreme values of a sample
Let $X_1, \ldots, X_n$ be independent random variables having a common distribution function $F$.

Let $M_n$ be the maximum of $n$ observations:

$$
M_n = \max \{ X_1, \ldots, X_n \}
$$

The theoretical distribution of $M_n$

$$
\Pr \{M_n \leq z\} = \Pr \{X_1 \leq z, \ldots, X_n \leq z\} = \Pr \{X_1 \leq z\} \times \cdots \times \Pr \{X_n \leq z\} = \{F(z)\}^n.
$$

not very useful since usually $F$ is unknown...
Basics of extreme value theory II

Make a simple linear renormalisation of $M_n$

\[ M_n^* = \frac{M_n - b_n}{a_n} \]

for constants \( \{a_n > 0\} \) and \( \{b_n\} \) chosen appropriately to stabilise the location and scale of \( M_n^* \) as \( n \) increases.
Theorem (Fréchet 1927): If there exist sequences of constants \( \{a_n > 0\} \) and \( \{b_n\} \) such that
\[
\Pr \left\{ \frac{M_n - b_n}{a_n} \leq z \right\} \rightarrow G(z) \quad \text{as } n \rightarrow \infty,
\]
where \( G \) is a non-degenerate distribution function, then \( G \) belongs to one of the following families:

I: \[ G(z) = \exp \left\{ - \exp \left[ - \left( \frac{z - b}{a} \right) \right] \right\}, \quad -\infty < z < \infty; \]

II: \[ G(z) = \begin{cases} 
0, & z \leq b, \\
\exp \left\{ - \left( \frac{z-b}{a} \right)^{-\alpha} \right\}, & z > b;
\end{cases} \]

III: \[ G(z) = \begin{cases} 
\exp \left\{ - \left[ - \left( \frac{z-b}{a} \right)^{\alpha} \right] \right\}, & z < b, \\
1, & z \geq b,
\end{cases} \]

independently of the underlying \( F \). (REM: central limit theorem for sample means)
• The Gumbel distribution
  \[ G(x) = \exp(-\exp(-x)) \]

• The Fréchet distribution
  \[ G(x) = \begin{cases} 
0 & x \leq 0 \\
\exp(-x-\alpha) & x > 0, \alpha > 0 
\end{cases} \]

• The Weibull distribution
  \[ G(x) = \begin{cases} 
\exp\left(-(-x)^\alpha\right) & x < 0, \alpha > 0 \\
1 & x \geq 0 
\end{cases} \]
Corollary: If there exist sequences of constants \( \{a_n > 0\} \) and \( \{b_n\} \) such that

\[
\Pr \left\{ \frac{M_n - b_n}{a_n} \leq z \right\} \to G(z) \quad \text{as } n \to \infty,
\]

for a non-degenerate distribution function \( G \), then \( G \) is a member of the GEV family

\[
G(z) = \exp \left\{ - \left[ 1 + \xi \left( \frac{z - \mu}{\sigma} \right) \right]^{-1/\xi} \right\}, \tag{2.2}
\]

defined on \( \{z : 1 + \xi (z - \mu) / \sigma > 0\} \), where \(-\infty < \mu < \infty\), \( \sigma > 0 \) and \(-\infty < \xi < \infty\).

\[
z_p = \begin{cases} 
\mu - \frac{\sigma}{\xi} \left[ 1 - \{-\log(1 - p)\}^{-\xi}\right], & \text{for } \xi \neq 0, \\
\mu - \sigma \log \{-\log(1 - p)\}, & \text{for } \xi = 0,
\end{cases}
\]

where \( G(z_p) = 1 - p \).

Definition: The extreme quantile \( z_p = G^{-1}(1 - p) \), where \( G \) is the distribution function of \( M_n \), is called the return level associated with the return period \( 1/p \).
$\mu$: location
$\sigma$: scale
$\xi$: slope
$T$-axis is transformed such that Gumbel-Distribution is a straight line.

$$F(x) = GEV(x; \mu, \sigma, \xi)$$ estimated CDF

$$Y(x) = -\log(-\log(F(x)))$$ Gumbel Variate

Horizontal axis is linear in $Y$.

$$T(x) = 1/(1 - F(x))$$ Return period

$x_k \quad k = 1, \ldots, N$ Block Maxima

$$\tilde{T}_k = \frac{N + 1}{N + 1 - \text{rank}(x_k)}$$ plotting points of block maxima $x_k$
Parametric resampling and confidence intervals
### Extreme tides: temporal series

<table>
<thead>
<tr>
<th>Location</th>
<th>High Water Levels</th>
<th>Monthly Matching Possibilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>9. Port-Tudy</td>
<td>194 tides &gt;= 551 cm</td>
<td>45.08%</td>
</tr>
<tr>
<td>10. Le Crouesty</td>
<td>15 tides &gt;= 579 cm</td>
<td>28.33%</td>
</tr>
<tr>
<td>11. Concarneau</td>
<td>33 tides &gt;= 540 cm</td>
<td>0%</td>
</tr>
<tr>
<td>12. Brest</td>
<td>1229 tides &gt;= 748 cm</td>
<td>30.26%</td>
</tr>
</tbody>
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<td>9. Port-Tudy</td>
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<tr>
<td>10. Le Crouesty</td>
<td>12 surges &gt;= 51 cm</td>
<td>28.33%</td>
</tr>
<tr>
<td>11. Concarneau</td>
<td>32 surges &gt;= 72 cm</td>
<td>0%</td>
</tr>
<tr>
<td>12. Brest</td>
<td>1186 surges &gt;= 63 cm</td>
<td>30.26%</td>
</tr>
</tbody>
</table>

Pirazzoli & Tomasin (2007) Ocean Dynamics 57, 91
Expected return periods

Pirazzoli & Tomasin (2007) Ocean Dynamics 57, 91
Basics of extreme value theory IV

Estimation of GEV parameters:

★ Method of moments: not robust and VERY unstable
★ **Block maxima**: extreme events in FIXED intervals
★ **Peak over thresholds**: all extreme events ABOVE a fixed value
Conditional excess distribution function for a threshold level $u$:

$$F_u(y) = P(y \geq X-u \mid X > u), \quad 0 \geq y < \infty,$$

$$F_u(y) = \frac{F(u + y) - F(u)}{1 - F(u)}, \quad y > 0.$$

Theorem (Pickands 1975):

$$F_u(y) \approx \begin{cases} 1 - \left(1 + \frac{\xi}{\sigma}y\right)^{-1/\xi} & \text{if } \xi \neq 0 \\ 1 - e^{-y/\sigma} & \text{if } \xi = 0 \end{cases}$$

so that the cumulative DF for events above $u$, taking $x = u + y$, is:

$$F(x) = 1 - \frac{N_u}{n} \left[1 + \frac{\xi}{\sigma}(x - u)\right]^{-1/\xi}$$
For a given sample, calculate the likelihood and apply standard Bayesian estimation:

Star Formation and the Origins of the Stellar Initial Mass Function
Why are there fewer massive stars than low-mass stars?
Typical stellar mass $\sim$ Jeans Mass.

$$M_{\text{Jeans}} \equiv \left[ \frac{5}{2} \frac{R_g T}{G \mu} \right]^\frac{3}{2} \left[ \frac{4\pi}{3} \rho \right]^{-\frac{1}{2}}.$$

In Molecular clouds:

- Temperatures $\sim 10$ K,
- Lots of structure,
- Dense cores: $\rho \sim 10^{-19}$ g/cm$^3$.

$\implies$ masses $M \sim 0.7$ $M_{\text{Sol}}$.

Agrees well with observations:

- $M_{\text{median}} \sim 0.5$ $M_{\text{Sol}}$.

Not all stars have the same mass $\implies$ Distribution?

The IMF!
- **Salpeter-IMF** (power-law): \( \xi(m) = k \, m^{-2.35} \), originally only for 0.4 to 10 \( M_{\odot} \).

- **Miller-Scalo IMF** (log-normal):

\[
\xi(lm) = k \, e^{-\frac{(lm+1.02)^2}{0.9248}},
\]

with \( lm = \log_{10} m \).

- **Kroupa-IMF** (multi power-law):

\[
\xi(m) = k \left\{ \begin{array}{ll}
\left( \frac{m}{m_H} \right)^{-\alpha_0} & , m_{\text{low}} \leq m < m_H, \\
\left( \frac{m}{m_H} \right)^{-\alpha_1} & , m_H \leq m < m_0, \\
\left( \frac{m}{m_0} \right)^{-\alpha_1} \left( \frac{m}{m_0} \right)^{-\alpha_2} & , m_0 \leq m < m_1, \\
\left( \frac{m}{m_0} \right)^{-\alpha_1} \left( \frac{m}{m_1} \right)^{-\alpha_2} \left( \frac{m}{m_1} \right)^{-\alpha_3} & , m_1 \leq m < m_{\text{max}},
\end{array} \right.
\]

\( \alpha_0 = +0.30 \) \hspace{1cm} 0.01 \leq m/M_\odot < 0.08, \\
\alpha_1 = +1.30 \hspace{1cm} 0.08 \leq m/M_\odot < 0.50, \\
\alpha_2 = +2.35 \hspace{1cm} 0.50 \leq m/M_\odot < 1.00, \\
\alpha_3 = +2.35 \hspace{1cm} 1.00 \leq m/M_\odot. \)
The diagram represents the Kroupa IMF (Initial Mass Function) distribution, where the vertical axis is \( \log_{10}(n(m) \ln(10)) \) and the horizontal axis is \( \log_{10}(m/M_\odot) \). Key features include:

- \( N: 92.3\% \), \( M: 44.4\% \)
- Lock-up fraction: 7.4%
- IMS: 36.7%
- SN II: 0.3%

The graph illustrates the distribution of stellar masses with percentages and specific mass ranges.
Evidence for an upper mass cutoff around $150 \, M_{\odot}$?
One expects a correlation between the total cluster mass and the maximum observed stellar mass.
Posterior PDFs for scale and slope of IMF for massive stars

Valls-Gabaud & Asensio Ramos (2012)
Distribution function of brightest galaxies in clusters

Bhavsar & Barrow (1985) MNRAS 213, 857
Probability of finding the largest structures in a cosmological volume
Probability of finding a Shapley supercluster

\[ R_{\text{Shapley}} = 31 \ h^{-1}\text{Mpc} \]

\[ \sigma_8 = 0.9 \]

Sheth & Diaferio (2011) MNRAS 417, 2938
Probability of finding the SDSS Great Wall

\[
V_{\text{SDSS Wall}} = 7.2 \times 10^5 \, h^{-3} \text{Mpc}^3
\]

\[
\sigma_8 = 0.8
\]

Sheth & Diaferio (2011) MNRAS 417, 2938
PDF of the most massive clusters of galaxies as a function of redshift and equation of state of Dark Energy

Waizmann, Ettori & Moscardini (2011) MNRAS 418, 456
Most massive galaxy clusters expected in a given survey

XMMU J2235.3-2557: a massive cluster
$M_{324} = (6.4 \pm 1.2) \times 10^{14} \, M_{\odot}$ at $z=1.4$

Cayon, Gordon & Silk (2011) MNRAS 415, 849
.... but the shape parameter of the EVS of the halo mass function does not discriminate non-gaussianity
Testing General Relativity

Abundance

\[ n(M, z) = \int_0^M f(\sigma) \frac{\bar{\rho}_m}{M'} \frac{d \ln \sigma^{-1}}{dM'} dM' \]

\[ \sigma^2(M, z) = \frac{1}{2\pi^2} \int_0^\infty k^2 P(k, z)|W_M(k)|^2 dk \]

\[ P(k, z) \propto k^{ns} T^2(k, z_t) D(z)^2 \]

\[ D(z) \equiv \frac{\delta(z)}{\delta(z_t)} \]

\[ \frac{d \ln \delta}{d \ln a} = \Omega_m(a)^\gamma \]

\[ \Omega_m(a) = \Omega_m a^{-3} / E(a)^2 \]

\[ \gamma \sim 0.55 \]

\[ E(a) = \left[ \Omega_m a^{-3} + \Omega_{de} a^{-3(1+w)} + \Omega_k a^{-2} \right]^{1/2} \]
Searching for strong gravitational lenses
Gravitational lens effect by a massive cluster
The size of the Einstein radius can be measured directly from the positions of the background galaxies. It can also provide us with the depth of the potential well (i.e. dark matter content) as well as the cosmological parameters of a given metric if the redshifts of the background galaxies can also be measured:

$$\theta_E = 4\pi \left( \frac{\sigma_{SIS}}{c} \right)^2 \frac{D_{LS}}{D_{OS}}$$

- positions of the images
- spectrum of the lens
- spectrum of the source
Inferring Einstein radii for 10,000 SDSS clusters

Zitrin et al. (2011) arXiv:1105.2295
Theoretical expectations

Oguri & Blandford (2009)
MNRAS 392, 930
Tension with \( \Lambda \)CDM scenario?

Zitrin et al. (2011) arXiv:1105.2295
Summary

Methodology

• Proper statistical theory for extreme events
• Bayesian thresholding estimates work best
• Wide applications to astrophysics and cosmology

Limitations

• Assessment of selection biases in samples
• May require extensive testing with simulations